Structural Stability and Partial Hyperbolicity in Large Dynamical Systems

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7/22/03

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Outline

- Initial motivation
- Preliminary notions and definitions
- Background and explanation of the problem
- Outline of arguments
- Arguments
- Other directions of work and summary
Source of motivation

- Bifurcation diagram and largest Lyapunov exponent for $n = 4$ and $d = 64$
• Bifurcation Diagram for $n = 64$ and $d = 64$

![Bifurcation Diagram](image)

• Lyapunov exponent for $n = 64$ and $d = 64$

![Lyapunov Exponent](image)
Preliminary notions

• Stable ($E^s$), unstable ($E^u$), and center manifolds ($E^c$): defined with respect to fixed points, define boundaries between different orbit types, define the geometry of orbits.

• Pictures
• Topological conjugacy:

**Definition 1 (Topological conjugacy)** Consider two \( C^r \) diffeomorphisms \( f : \mathbb{R}^n \to \mathbb{R}^n \) and \( g : \mathbb{R}^n \to \mathbb{R}^n \). \( f \) and \( g \) are said to be \( C^k \) \((k \geq r)\) conjugate if there exists a \( C^k \) diffeomorphism such that \( g = h \circ g \circ h^{-1} \). If \( k = 0 \), \( f \) and \( g \) are said to be topologically conjugate.

• Structural stability:

**Definition 2 (Structural Stability)** A \( C^r \) discrete time map, \( f \), is structurally stable if there is a \( C^r \) neighborhood, \( V \), such that any \( g \in V \) is topologically conjugate to \( f \), i.e. there exists a homeomorphism \( h \) such that \( f = h^{-1} \circ g \circ h \).

• Pictures

• Perturbation: three types - functional form, parameter variation, initial condition variation
• Hyperbolicity:

**Definition 3 (Hyperbolic linear map)** A linear map of $\mathbb{R}^n$ is called hyperbolic if all of its eigenvalues are different from one.

**Definition 4 (Hyperbolic periodic point)** $p$ is a hyperbolic periodic point for $f$ if $(Df^n)_p : T_p M \to T_p M$ is a hyperbolic linear map. It’s orbit will be called a hyperbolic periodic point.

**Definition 5 (Hyperbolic map)** A discrete time map $f$ is said to be hyperbolic on a compact invariant set $\Lambda$ if there exists a continuous splitting of the tangent bundle, $TM|_\Lambda = E^s \oplus E^u$, and there are constants $C > 0$, $0 < \lambda < 0$, such that $\|Df^n|_{E^s_x}\| < C\lambda^n$ and $\|Df^{-n}|_{E^u_x}\| < C\lambda^n$ for any $n > 0$ and $x \in \Lambda$.

where the stable manifold $E^s$ [respectively unstable $E^u$] of $x \in \Lambda$ is the set of points $p \in M$ such that $|f^k(x) - f^k(p)| \to 0$ as $k \to \infty$. 
Lyapunov exponents: correspond to stable, unstable, and center manifolds of an orbit; measure rates of expansion and contraction

**Definition 6 (Lyapunov Exponents)** Define the discrete dynamical system \( f : \mathbb{R}^n \to \mathbb{R}^n \) and a point in the domain, \( x \in \mathbb{R}^n \). Suppose there are subspaces \( V_i^{(1)}V_i^{(2)} \cdots V_i^{(n)} \) in the tangent space of \( f^i(x) \) and scalars \( \chi_1 \leq \chi_2 \leq \cdots \leq \chi_n \) with the properties:

1. \( Df(V_i^{(j)}) = V_{i+1}^{(j)} \)
2. \( \dim V_i^{(j)} = n + 1 - j \)
3. \( \lim_{N \to \infty} \ln \| \sqrt{(Df^N)^T(Df^N)} \cdot v \| = \chi_j \) for all \( v \in V_0^{(j)} - V_0^{(j+1)} \)

The numbers \( \chi_j \) are called the Lyapunov exponents of \( f \) at \( x \).

\[
\chi_j = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \ln(< (Df_k \cdot \delta x_j)^T, (Df_k \cdot \delta x_j) >)
\]

where \(<, >\) is the standard inner product, \( \delta x_j \) is the \( j^{th} \) component of the \( x \) variation and \( Df_k \) is the “orthogonalized” Jacobian of \( f \) at the \( k^{th} \) iterate of \( f(x) \).
• Non-wandering set:

**Definition 7 (Non-wandering set)** A point \(x_0\) is called non-wandering if the following holds for any neighborhood \(U\) of \(x_0\) for some \(n \neq 0\):

\[ g^n(U) \cap U \neq 0 \] (2)

The set of all such points is called the non-wandering set.

• Attractor (or orbit):

**Definition 8 (Attractor)** A closed invariant set \(\Lambda \subset \mathbb{R}^n\) is called an attracting set if there is some neighborhood \(U\) of \(\Lambda\) such that:

\[ g^n(x) \in U \text{ and } g^n(x) \to \Lambda \text{ as } n \to \infty \] (3)

• Dense periodic orbits: The given the invariant set (attractor) \(\Lambda\) has dense (maybe not stable) periodic orbits.

• Strong transversality: YIKES - \(M_x = E_x^s + E_x^u\) for all \(x \in M\) - i.e. a continuous “splitting” of the manifolds into \(E^s\) and \(E^u\)
• Axiom $A$:

**Definition 9 (Axiom A)** $A C^r f$ satisfies axiom $A$ if and only if $\Omega$ is hyperbolic and the periodic points of $f$ are dense.

• Structurally stable dynamical systems satisfy axiom $A$ (most importantly are hyperbolic):

**Theorem 1 (Mane - theorem $A$)** Every $C^1$ structurally stable diffeomorphism of a closed manifold satisfies Axiom $A$.

• Axiom $A$ and strong transversality guarantees structural stability:

**Theorem 2 (Robbin - Structural Stability Theorem)**

A $C^2$ diffeomorphism (on a compact, boundaryless manifold) which satisfies axiom $A$ and the strong transversality conditions is structurally stable.
- Partially hyperbolic dynamical systems

**Definition 10 (Partial hyperbolicity)** The discrete time map \( f \) is said to be partially hyperbolic if the tangent bundle \( TM \) splits as a \( T^f \)-invariant sum:

\[
TM = E^U \oplus E^C \oplus E^S
\]

where at least two of the sub bundles are non-trivial, and four constants, \( a < b < 1 < c < d \), and a Finsler structure \( || \cdot || \) on \( M \) such that for all \( x \in M \), and all \( v \in T_x M \):

\[
v \in E^U(x) \Rightarrow d||v|| \leq ||T_x f v|| \quad (5)
v \in E^C(x) \Rightarrow b||v|| \leq ||T_x f v|| \leq c||v|| \quad (6)
v \in E^S(x) \Rightarrow ||T_x f v|| \leq a||v|| \quad (7)
\]

Where \( E^U \), \( E^S \), \( E^C \) are the unstable, stable and center bundles for \( f \), and a Finsler structure on the tangent bundle can be defined:

**Definition 11 (Finsler structure)** A Finsler structure on the tangent bundle of a Banach manifold \( M \) is a continuous function \( || \cdot || : TM \to [0, \infty) \) such that:

(i) For every \( x \in M \), the restriction \( || \cdot ||_x = || \cdot |||_{T_x M} \) is an equivalent norm on the tangent space \( T_x M \),
(ii) For every $x_0 \in M$, and $k > 1$, there is a trivializing neighborhood $U$ of $x_0$ within which

$$\frac{1}{k}||\cdot||_x \leq ||\cdot||_{x_0} \leq k||\cdot||_x$$  \hspace{2cm} (8)$$

A $C^1$ Banach manifold $M$ together with a Finsler structure on its tangent bundle is said to be a Finsler manifold.
Intuition

- Assume $C^r$ one parameter discrete time maps transforming bounded subsets of $\mathbb{R}^d$ to themselves
Outline of arguments: i.e. the ingredients

• As $d$ increases we need:
  – increasing continuity of Lyapunov exponents
  – increase in the maximum number of Lyapunov exponents
  – decrease in the distance between exponent zero crossings

• With the above trends, given arbitrarily high $d$, we can find a subset in parameter space such that we can approximate violation of the stability conjecture

• Difference between strict mathematics and computational or experimental observation
Numerical arguments: i.e. making sure we are seeing what we think we are seeing

- Error in Lyapunov exponent calculation:
- Continuity of Lyapunov exponents
• Increase in the number of exponents
- Decrease in the distance between exponent zero crossings
• Upon varying our parameter we see:
  
  – Hyperbolicity violation on an “increasingly” dense - yet not open and Lebesque measure zero - set.
  
  – I.e. we can find a subset of parameter space such that as the dimension is increased, the “chance” of topological change versus “small” parameter variation becomes small (zero codimension bifurcation volume)
  
  – With increasing dimension, very low probability of periodic windows in certain subsets of parameter space
Future work

- General: understand the effects of perturbation of initial conditions and $d$ parameter variation

- Achieve a better understanding of the seemingly robust nature of chaos and the non-existence of periodic windows - specifically Yorke’s windows conjecture.

- Achieve a better understanding of the basins of attraction - existence of Milnor attractors, effects of SRB measures, and partial hyperbolicity.

- Achieve an understanding of the route out of chaos - the “high s” limit